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The flow near the edge of a disc at rest in a rotating fluid

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Summary

The Reynolds number Re being based on the angular velocity of the fluid and the radius of the disc, it is shown that within a distance $O(Re^{-2/3})$ from the edge of the disc, the flow is determined by the Navier-Stokes equations. The boundary-value problem describing this flow is formulated. The asymptotic behaviour of its solution is investigated analytically and its complete numerical solution is evaluated. Results for various physical quantities, among them the additional torque due to the Navier-Stokes flow, are presented.

1. Introduction

The classical problem of the flow of a rotating fluid above an infinite plane at rest has been considered by Bödewadt [1]. This leads to a system of three coupled ordinary differential equations for the three velocity components, which can be solved numerically with arbitrary accuracy. The solution obtained by Browning is given in Schlichting's book [2]. If the plane is replaced by a circular disc of finite radius, the problem becomes much more complicated.

On the basis of boundary-layer theory the problem of the finite disc has been considered by Stewartson [3], Rogers and Lance [4] and by Belcher, Burggraf and Stewartson [5]. They give expansions of the solution near the edge of the disc. However, these are not valid in the immediate vicinity of the edge since the boundary-layer equations lose their validity there. Due to the different boundary conditions at the disc and just outside the disc, it is necessary to use the Navier-Stokes equations there. Since there is inward flow near the edge of the disc, the problem bears resemblance to the leading-edge problem of a flat plate placed in a uniform flow, [6]. However, it is more complicated in several respects.

It is shown in the present paper that the Navier-Stokes region near the edge of the disc is $O(Re^{-2/3})$, where $Re = \Omega a^2/\nu$ with Ω the angular velocity, a the radius of the disc and ν the kinematic viscosity. The modification of the boundary-layer solution to the Navier-Stokes solution gives rise to an additional term in the expression for the torque acting on the disc, which is $O(Re^{-1})$. The torque itself is $O(Re^{-1/2})$.

It may be remarked that the flow near the edge of a rotating disc in a fluid at rest (von Karman problem) resembles the trailing-edge problem of a flat plate and hence will show a multiple-deck structure.

2. The Navier-Stokes region (size and equations)

The full Navier-Stokes equations for the rotating fluid are in dimensionless form as follows, see e.g. [2],

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{\partial p}{\partial r}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right),$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\frac{1}{Re}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right),$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right),$$

with the continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}(ur)+\frac{\partial w}{\partial z}=0.$$

Lengths have been made dimensionless with a, velocities with Ωa and the pressure with $\rho\Omega^2 a^2$.

Introduction of a stream function ψ by

$$u=-\frac{1}{r}\frac{\partial\psi}{\partial z},\qquad w=\frac{1}{r}\frac{\partial\psi}{\partial r},$$

introduction of a tangential vorticity component γ by

$$\gamma=\frac{\partial w}{\partial r}-\frac{\partial u}{\partial z}$$

and elimination of the pressure leads to the following system of equations for v, γ and ψ:

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\frac{1}{Re}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right), \tag{1a}$$

$$u\frac{\partial \gamma}{\partial r}+w\frac{\partial \gamma}{\partial z}-\frac{u\gamma}{r}+\frac{2v}{r}\frac{\partial v}{\partial z}=\frac{1}{Re}\left(\frac{\partial^2 \gamma}{\partial r^2}+\frac{1}{r}\frac{\partial \gamma}{\partial r}-\frac{\gamma}{r^2}+\frac{\partial^2 \gamma}{\partial z^2}\right), \tag{1b}$$

$$\gamma=\frac{1}{r}\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}\right). \tag{1c}$$

We now suppose the Navier-Stokes region to be of size $Re^{-\alpha}$ and the stream function in that region to be $O(Re^{-\beta})$, where α and β are both positive. Thus

$$1-r\sim O(Re^{-\alpha}),\qquad z\sim O(Re^{-\alpha}),\qquad \psi\sim O(Re^{-\beta}).$$

This means:

u and w are $O(Re^{\alpha-\beta})$, $\quad \gamma \sim O(Re^{2\alpha-\beta})$,

1st derivatives of γ are $O(Re^{3\alpha-\beta})$,

2nd derivatives of γ are $O(Re^{4\alpha-\beta})$.

Since the Navier-Stokes region must be matched to the boundary-layer solution for small positive values of $1-r$ and to the rotating flow at the rest of its boundary, we have

$v \sim O(1)$,

1st derivatives of v are $O(Re^{\alpha})$,

2nd derivatives of v are $O(Re^{2\alpha})$.

The most important terms at the left-hand side of Eqn. (1a) are $O(Re^{2\alpha-\beta})$, while the most important terms at the right-hand side are $O(Re^{2\alpha-1})$. These terms must be of the same order and hence $\beta = 1$.

Since in Eqn. (1b) the term $2vr^{-1}\partial v/\partial z$ is the term which causes the secondary flow (γ and ψ to be different from zero), this term, which is $O(Re^{\alpha})$, must belong to the most important terms in this equation. The other most important terms are $O(Re^{4\alpha-2\beta})$ and $O(Re^{4\alpha-\beta-1})$, which both are $O(Re^{4\alpha-2})$. Hence

$$\alpha = 4\alpha - 2 \Rightarrow \alpha = \tfrac{2}{3}.$$

This means that the size of the Navier-Stokes region is $O(Re^{-2/3})$.

In this region we introduce the following quantities of $O(1)$, denoted by capitals,

$$\begin{aligned}
&\Psi = Re\psi,\\
&X = Re^{2/3}(1-r), \qquad Z = Re^{2/3}z,\\
&U = Re^{1/3}u, \qquad W = Re^{1/3}w, \qquad V = v,\\
&\Gamma = Re^{-1/3}\gamma, \qquad \frac{\partial \Gamma}{\partial X} = -Re^{-1}\frac{\partial \gamma}{\partial r}, \qquad \frac{\partial \Gamma}{\partial Z} = Re^{-1}\frac{\partial \gamma}{\partial z},\\
&\frac{\partial^2 \Gamma}{\partial X^2} = Re^{-5/3}\frac{\partial^2 \gamma}{\partial r^2}, \qquad \frac{\partial^2 \Gamma}{\partial Z^2} = Re^{-5/3}\frac{\partial^2 \gamma}{\partial z^2},\\
&\frac{\partial V}{\partial X} = -Re^{-2/3}\frac{\partial v}{\partial r}, \qquad \frac{\partial V}{\partial Z} = Re^{-2/3}\frac{\partial v}{\partial z},\\
&\frac{\partial^2 V}{\partial X^2} = Re^{-4/3}\frac{\partial^2 v}{\partial r^2}, \qquad \frac{\partial^2 V}{\partial Z^2} = Re^{-4/3}\frac{\partial^2 v}{\partial z^2}.
\end{aligned} \tag{2}$$

Substituting into Eqns. (1) and retaining only the most important terms, which implies to take $r = 1$, we come to the following system of equations:

$$\frac{\partial \Psi}{\partial Z}\frac{\partial V}{\partial X} - \frac{\partial \Psi}{\partial X}\frac{\partial V}{\partial Z} = \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Z^2}, \tag{3a}$$

$$\frac{\partial \Psi}{\partial Z}\frac{\partial \Gamma}{\partial X} - \frac{\partial \Psi}{\partial X}\frac{\partial \Gamma}{\partial Z} + 2V\frac{\partial V}{\partial Z} = \frac{\partial^2 \Gamma}{\partial X^2} + \frac{\partial^2 \Gamma}{\partial Z^2}, \tag{3b}$$

$$\Gamma = \frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Z^2}, \tag{3c}$$

where it has been used that

$$U = -\frac{\partial \Psi}{\partial Z} \quad \text{and} \quad W = -\frac{\partial \Psi}{\partial X}.$$

The boundary conditions are at the disc,

$$X > 0, \qquad Z = 0: \qquad \Psi = 0, \qquad \frac{\partial \Psi}{\partial Z} = 0, \qquad V = 0, \tag{4a}$$

in the symmetry plane outside the disc,

$$X < 0, \qquad Z = 0: \qquad \Psi = 0, \qquad \frac{\partial V}{\partial Z} = 0, \qquad \Gamma = 0, \tag{4b}$$

since V is even in Z, while Ψ and Γ are odd functions of Z. Due to symmetry we need only to consider the half-plane $Z \geqslant 0$.

For $X \to \infty$ the solution must become identical to the boundary-layer solution given in [3] and [4], which is

$$\psi = Re^{-1/2}(1-r)^{3/4}\{\psi_0(\tau) + O(1-r)\},$$

$$v + V_0(\tau) + O(1-r),$$

where $\tau = Re^{1/2}z(1-r)^{-1/4}$.

By aid of Eqns. (2) this is transformed to

$$X \to \infty, \qquad \Psi = X^{3/4}\psi_0(\tau), \qquad V = V_0(\tau), \qquad \tau = ZX^{-1/4}. \tag{4c}$$

The order terms have been omitted since they are $O(Re^{-2/3})$ smaller.

The functions $V_0(\tau)$ and $\psi_0(\tau)$ satisfy the differential equations

$$V_0'' + \tfrac{3}{4}\psi_0 V_0' = 0,$$

$$\psi_0''' + \tfrac{3}{4}\psi_0\psi_0'' - \tfrac{1}{2}\psi_0'^2 = V_0^2 - 1 \tag{5}$$

with boundary conditions

$$V_0(0)=0, \qquad V_0(\infty)=1,$$

$$\psi_0(0)=0, \qquad \psi_0'(0)=0, \qquad \psi_0'(\infty)=0. \tag{6}$$

Its solution has the following values

$$V_0'(0)=0.439847747,$$

$$\psi_0''(0)=1.068126931,$$

$$\psi_0(\infty)=1.691543111.$$

Finally, for $Z \to \infty$ and for $X \to -\infty$, the solution must represent the flow outside the viscous regions. There $V=1$, $\Gamma=0$ while Ψ must be matched from its value $X^{3/4}\psi_0(\infty)$ in (4c) to $\Psi=0$ as in (4b).

3. Transformation to other coordinates

Two different coordinate systems will be used, one for the analytical work and one for the numerical procedure. First, we introduce parabolic coordinates ξ and η by

$$X+iZ=(\xi+i\eta)^2, \qquad X=\xi^2-\eta^2, \qquad Z=2\xi\eta. \tag{7}$$

Transformation of Eqns. (3) to these coordinates results in

$$\frac{\partial\Psi}{\partial\eta}\frac{\partial V}{\partial\xi}-\frac{\partial\Psi}{\partial\xi}\frac{\partial V}{\partial\eta}=\frac{\partial^2 V}{\partial\xi^2}+\frac{\partial^2 V}{\partial\eta^2}, \tag{8a}$$

$$\frac{\partial\Psi}{\partial\eta}\frac{\partial\Gamma}{\partial\xi}-\frac{\partial\Psi}{\partial\xi}\frac{\partial\Gamma}{\partial\eta}+4V\left(\eta\frac{\partial V}{\partial\xi}+\xi\frac{\partial V}{\partial\eta}\right)=\frac{\partial^2\Gamma}{\partial\xi^2}+\frac{\partial^2\Gamma}{\partial\eta^2}, \tag{8b}$$

$$4(\xi^2+\eta^2)\Gamma=\frac{\partial^2\Psi}{\partial\xi^2}+\frac{\partial^2\Psi}{\partial\eta^2}. \tag{8c}$$

In parabolic coordinates only the quarter-plane $\xi \geqslant 0$, $\eta \geqslant 0$ needs to be considered.
The boundary conditions along the coordinate axes are

$$\xi>0, \qquad \eta=0: \qquad \Psi=0, \qquad \frac{\partial\Psi}{\partial\eta}=0, \qquad V=0, \tag{9a}$$

$$\xi=0, \qquad \eta>0: \qquad \Psi=0, \qquad \Gamma=0, \qquad \frac{\partial V}{\partial\xi}=0. \tag{9b}$$

Near the origin we have Stokes flow, where inertia forces can be neglected. The equations then become

$$0=\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2},$$

$$0=\frac{\partial^2 \Gamma}{\partial \xi^2}+\frac{\partial^2 \Gamma}{\partial \eta^2},$$

$$4(\xi^2+\eta^2)\Gamma=\frac{\partial^2 \Psi}{\partial \xi^2}+\frac{\partial^2 \Psi}{\partial \eta^2}.$$

The solution of this set of equations which satisfies the boundary conditions is

$$\Psi=2A\xi\eta^2, \qquad \Gamma=A\frac{\xi}{\xi^2+\eta^2}, \qquad V=B\eta, \tag{10}$$

where A and B are arbitrary constants. For Ψ and Γ this solution is identical to the Carrier-Lin solution near the leading edge of the flat plate, see [6] and [7]. It can be verified with the aid of Eqns. (8) that all neglected inertia terms are of smaller order of magnitude than the terms retained.

The boundary conditions for $\xi \to \infty$ become

$$\Psi=\xi^{3/2}\psi_0(\tau), \qquad V=V_0(\tau) \quad \text{where} \quad \tau=2\xi^{1/2}\eta. \tag{9c}$$

Due to this condition ξ and τ would be more suitable coordinates in the viscous region than ξ and η. However, for $\xi=0$ and finite values of τ, η goes to infinity. Therefore, it is better to take

$$\tau_1=2\sqrt{1+\xi}\cdot\eta, \tag{11}$$

which has the same character for $\xi \to \infty$ as $\tau=2\xi^{1/2}\eta$. Taking also in account Eq. (8c) and condition (9c) the boundary conditions for $\xi \to \infty$ become

$$\Psi=\xi^{3/2}\psi_0(\tau_1)-\tfrac{1}{2}\xi^{1/2}\tau_1\psi_0'(\tau_1), \qquad V=V_0(\tau_1)+O(\xi^{-1}) \quad \text{and}$$

$$\Gamma=\xi^{1/2}\psi_0''(\tau_1)+O(\xi^{-1/2}). \tag{9d}$$

The second system of coordinates, denoted by κ, λ, is obtained by a transformation similar to (7), namely

$$\xi+i\eta=(\kappa+i\lambda)^2, \qquad \xi=\kappa^2-\lambda^2, \qquad \eta=2\kappa\lambda. \tag{12}$$

Transformation of Eqns. (8) to the κ, λ-coordinates yields

$$\frac{\partial \Psi}{\partial \lambda}\frac{\partial V}{\partial \kappa}-\frac{\partial \Psi}{\partial \kappa}\frac{\partial V}{\partial \lambda}=\frac{\partial^2 V}{\partial \kappa^2}+\frac{\partial^2 V}{\partial \lambda^2}, \tag{13a}$$

$$\frac{\partial\Psi}{\partial\lambda}\frac{\partial\Gamma}{\partial\kappa}-\frac{\partial\Psi}{\partial\kappa}\frac{\partial\Gamma}{\partial\lambda}+8V\left\{(3\kappa^2-\lambda^2)\lambda\frac{\partial V}{\partial\kappa}+(\kappa^2-3\lambda^2)\kappa\frac{\partial V}{\partial\lambda}\right\}=\frac{\partial^2\Gamma}{\partial\kappa^2}+\frac{\partial^2\Gamma}{\partial\lambda^2}, \qquad (13b)$$

$$16(\kappa^2+\lambda^2)^3\Gamma=\frac{\partial^2\Psi}{\partial\kappa^2}+\frac{\partial^2\Psi}{\partial\lambda^2}. \qquad (13c)$$

The region of interest is now the sector of the κ, λ-plane between the lines $\lambda = 0$ and $\lambda = \kappa$ (argument between 0 and $\pi/4$).

The boundary conditions are

$$\kappa \geqslant 0, \quad \lambda = 0: \quad \Psi = 0, \quad \frac{\partial\Psi}{\partial\lambda} = 0, \quad V = 0, \qquad (14a)$$

$$\lambda = \kappa: \quad \Psi = 0, \quad \Gamma = 0, \quad \frac{\partial V}{\partial\kappa} - \frac{\partial V}{\partial\lambda} = 0, \qquad (14b)$$

$$\kappa \to \infty \quad \begin{cases} \Psi = \kappa^3\psi_0(\tau_2), & V = V_0(\tau_2), \\ \Gamma = \kappa\psi_0''(\tau_2), & \text{where } \tau_2 = 4\kappa^2\lambda. \end{cases} \qquad (14c)$$

The final values $\psi_0(\infty)$ and $V_0(\infty) = 1$ are approached exponentially. This means that these values are already approximated with great accuracy for a finite value of τ, say τ_b. For $\tau > \tau_b$ and $\xi \to \infty$ one has potential flow, that is $\Gamma = 0$ and $V = 1$, while Ψ is a harmonic function; τ_b was taken equal to 28.

In both the ξ, η-plane and the κ, λ-plane the region of interest is divided into 3 parts (see Fig. 1). Region I is the region where the full equations (8) and (13) must be

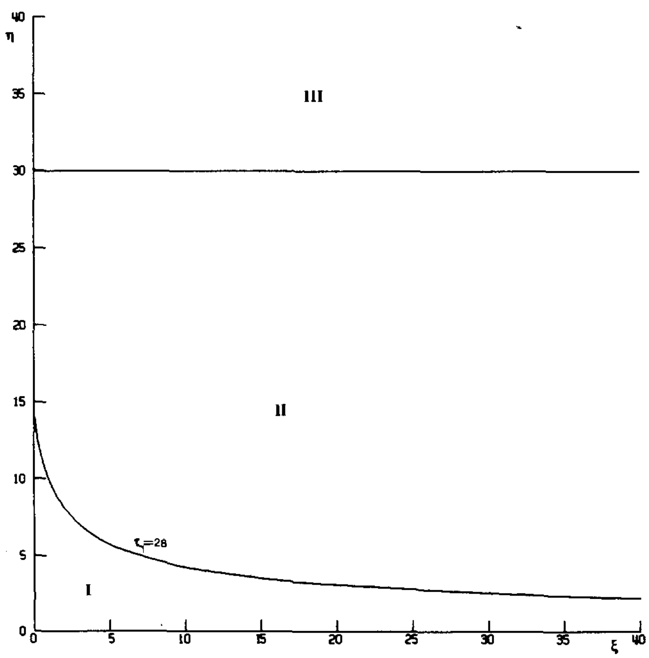

Figure 1. Division of the integration region.

considered. It is limited in the ξ, η-plane by the curve $2\sqrt{1+\xi} \cdot \eta = 28$ and in the κ, λ-plane by $4\kappa^2\lambda = 28$. Region II is limited in the ξ, η-plane by the line $\eta = 30$. Numerical calculations, which were performed in ξ, η-coordinates show that it is justified to take $V = 1$ and $\Gamma = 0$ in region II. In regions II and III only the harmonic function Ψ has to be determined. In region III no difference scheme for the numerical solution is necessary since the solution there can be given by aid of a Green's function.

4. The asymptotic behaviour at infinity

The asymptotic behaviour will be investigated in the κ, λ-plane. In region I the main term of the asymptotic behaviour is given by (14c). For region III we introduce polar coordinates r, θ in the κ, λ-plane. The asymptotic expansion of the harmonic function Ψ with $\Psi = 0$ for $\theta = \pi/4$ is

$$\Psi = Pr^3(\cos 3\theta + \sin 3\theta) + Qr^2 \cos 2\theta + Sr(\cos\theta - \sin\theta) + T\left(\frac{\pi}{4} - \theta\right), \tag{15}$$

where P, Q, S and T are constants to be determined. The highest power of r is 3 in agreement with the behaviour in region I.

A useful asymptotic expansion in region II is obtained by transformation of (15) to κ, λ-coordinates. This is

$$\Psi = P\kappa^3 + 3\kappa^2\lambda P - 3\kappa\lambda^2 P - P\lambda^3 + Q\kappa^2 - Q\lambda^2 + S\kappa - S\lambda + T\left(\frac{\pi}{4} - \tan^{-1}\frac{\lambda}{\kappa}\right). \tag{16}$$

This expression must be matched to the asymptotic behaviour in region I. For $\lambda = \tau_2/(4\kappa^2) \to 0$ we obtain

$$\Psi = P\kappa^3 + \tfrac{3}{4}P\tau_2 + Q\kappa^2 + S\kappa + \frac{\pi}{4}T. \tag{17}$$

Matching to (14c) yields $P = \psi_0(\infty)$. In order to determine Q, S and T we have to investigate what the further terms in the expansion in region I are.

Let 2 terms of the expansions for Ψ and V be

$$\Psi = \kappa^3\psi_0(\tau_2) + \kappa^{3-k}\psi_k(\tau_2),$$

$$V = V_0(\tau_2) + \kappa^{-k}V_k(\tau_2), \qquad k > 0. \tag{18}$$

The difference between τ_2 and $\tau = ZX^{-1/4}$ is $O(\kappa^{-6})$ for $\kappa \to \infty$. There will certainly be a second term for $k = 6$ but we are interested in values of k smaller than 6.

Substitution of Ψ in Eqn. (13c) leads to

$$\Gamma = \kappa\psi_0''(\tau_2) + \kappa^{1-k}\psi_k''(\tau_2).$$

Substitution of Ψ and V in Eqn (13a) gives as coefficient of κ^4 Eqn. (5a). The coefficient of κ^{4-k} leads to the equation linear in V_k and ψ_k,

$$4V_k'' + 3\psi_0 V_k' + k\psi_0' V_k + (3-k)V_0'\psi_k = 0. \tag{19}$$

Substitution of Ψ, V and Γ in Eqn. (13b) gives as coefficient of κ^5,

$$\psi_0'\psi_0'' - 3\psi_0\psi_0''' + 8V_0V_0' = 4\psi_0^{IV}.$$

This equation can once be integrated. Determining the constant of integration by using the boundary condition (6) for $\tau_2 \to \infty$, we find back Eqn. (5b).

The coefficient of κ^{5-k} leads to the linear equation

$$4\psi_k^{IV} + 3\psi_0\psi_k''' - (1-k)\psi_0'\psi_k'' - \psi_0''\psi_k' + (3-k)\psi_0'''\psi_k - 8(V_0V_k' + V_0'V_k) = 0.$$

Also this equation can once be integrated with the result that

$$4\psi_k''' + 3\psi_0\psi_k'' - (4-k)\psi_0'\psi_k' + (3-k)\psi_0''\psi_k - 8V_0V_k = 0, \tag{20}$$

where the boundary condition

$$\tau_2 \to \infty, \qquad \psi_k''(\tau_2) \to 0 \quad \text{and} \quad V_k(\tau_2) \to 0$$

has been used.

Now, the question is: can $\psi_k'(\infty)$ be different from 0, that is, can $\psi_k(\tau_2)$ be linear in τ_2 for $\tau_2 \to \infty$? If this were so, Ψ contains a second term $\kappa^{3-k}\tau_2$. Such term must be matched to Eqn. (17), valid in region II for $\lambda \to 0$. It is seen that this is impossible for $k < 3$ but required for $k = 3$.

Equation (19) then becomes

$$4V_3'' + 3(\psi_0V_3)' = 0$$

or, integrated,

$$4V_3' + 3\psi_0V_3 = 0$$

with the solution

$$V_3(\tau) = A\exp\left(-\tfrac{3}{4}\int_0^\tau \psi_0 \mathrm{d}\tau\right).$$

Since $V_3(0)$ should be zero, we find that $V_3(\tau)$ is identically zero.

Equation (20) for ψ_3 now becomes

$$4\psi_3''' + 3\psi_0\psi_3'' - \psi_0'\psi_3' = 0$$

with boundary conditions

$$\psi_3(0) = 0, \qquad \psi_3'(0) = 0, \qquad \psi_3'(\infty) = \tfrac{3}{4}\psi_0(\infty).$$

For $\tau_2 \to \infty$ we have $\psi_3(\tau_2) = \alpha\tau_2 + \beta$ with $\alpha = \frac{3}{4}\psi_0(\infty) = 1.268657333$. Numerical integration leads to

$$\beta = -2.35267532.$$

Returning to the asymptotic expressions (15), (16) and (17) it will be clear that

$$Q=0, \qquad S=0, \qquad T=\frac{4}{\pi}\beta. \tag{21}$$

5. The origin shift and the first eigenfunction

The asymptotic behaviour in region I,

$$\Psi \sim X^{3/4}\psi_0\left(\frac{Z}{X^{1/4}}\right), \quad V \sim V_0\left(\frac{Z}{X^{1/4}}\right) \quad \text{for} \quad X \to \infty,$$

would also be valid if the edge of the disc were not at $X=0$, but at some other finite value of X. Hence

$$\Psi \sim (X+a)^{3/4}\psi_0\left(\frac{Z}{(X+a)^{1/4}}\right), \qquad V \sim V_0\left(\frac{Z}{(X+a)^{1/4}}\right)$$

also describes the asymptotic behaviour. For small values of a the differences between the two expressions are terms with $\partial\Psi/\partial X$ and $\partial V/\partial X$. Hence $\partial\Psi/\partial X$ and $\partial V/\partial X$ must also be present in the asymptotic expansion of Ψ and V, respectively. We calculate these derivatives from the asymptotic behaviour, given above, as

$$\frac{\partial\Psi}{\partial X} = \frac{1}{4\kappa}(3\psi_0 - \tau_2\psi_0'), \qquad \frac{\partial V}{\partial X} = -\frac{1}{4\kappa^4}\tau_2 V_0' \quad \text{for} \quad \kappa \to \infty.$$

The asymptotic expansions (17) can now be extended as

$$\begin{aligned} \Psi &= \kappa^3\psi_0(\tau_2) + \psi_3(\tau_2) + \kappa^{-1}\psi_4(\tau_2), \\ \Gamma &= \kappa\psi_0''(\tau_2) + \kappa^{-2}\psi_3''(\tau_2) + \kappa^{-3}\psi_4''(\tau_2), \\ V &= V_0(\tau_2) + \kappa^{-4}V_4(\tau_2), \end{aligned} \tag{22}$$

where $\psi_4(\tau_2) = c(3\psi_0 - \tau_2\psi_0')$ and $V_4(\tau_2) = -c\tau_2 V_0'$ with c a constant, which cannot be determined from asymptotics but which follows from the complete solution of the flow field. The eigenfunctions $\psi_4(\tau_2)$ and $V_4(\tau_2)$ satisfy Eqns. (19) and (20) for $k=4$ and the boundary conditions

$$\psi_4(0)=0, \qquad \psi_4'(0)=0, \qquad \psi_4'(\infty)=0, \qquad V_4(0)=0, \qquad V_4(\infty)=0.$$

There exists no smaller value of k which admits a non-trivial solution of Eqns. (18) and (19) with homogeneous boundary conditions.

6. Numercial calculations (theory)

The numerical calculations have been performed in the ξ, η-plane since these coordinates are better adapted to the behaviour near the origin.

Since Γ goes to infinity near the origin, we introduce a new variable

$$K = (\xi^2 + \eta^2)\Gamma.$$

The equations (8) then become

$$\frac{\partial \Psi}{\partial \eta}\frac{\partial V}{\partial \xi} - \frac{\partial \Psi}{\partial \xi}\frac{\partial V}{\partial \eta} = \frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2}, \tag{24a}$$

$$\frac{\partial \Psi}{\partial \eta}\frac{\partial K}{\partial \xi} - \frac{\partial \Psi}{\partial \xi}\frac{\partial K}{\partial \eta} - \frac{2K}{\xi^2+\eta^2}\left(\xi\frac{\partial \Psi}{\partial \eta} - \eta\frac{\partial \Psi}{\partial \xi}\right) + 4(\xi^2+\eta^2)V\left(\eta\frac{\partial V}{\partial \xi} + \xi\frac{\partial V}{\partial \eta}\right)$$

$$+\frac{4}{\xi^2+\eta^2}\left(\xi\frac{\partial K}{\partial \xi} + \eta\frac{\partial K}{\partial \eta} - K\right) = \frac{\partial^2 K}{\partial \xi^2} + \frac{\partial^2 K}{\partial \eta^2}, \tag{24b}$$

$$4K = \frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2}, \tag{24c}$$

$$\xi > 0, \qquad \eta = 0: \qquad \Psi = 0, \qquad \frac{\partial \Psi}{\partial \eta} = 0, \qquad V = 0, \tag{25a}$$

$$\xi = 0, \qquad \eta > 0: \qquad \Psi = 0, \qquad K = 0, \qquad \frac{\partial V}{\partial \xi} = 0, \tag{25b}$$

$$\xi \to \infty, \qquad \tau_1 \text{ finite:} \qquad \Psi \sim \xi^{3/2}\psi_0(\tau_1), \qquad K \sim \xi^{5/2}\psi_0''(\tau_1). \tag{25c}$$

In *region I* we transform Eqns. (24) to new coordinates, defined by

$$\xi_1 = \xi, \qquad \tau_1 = \frac{\eta}{A(\xi)} \quad \text{with} \quad A(\xi) = \frac{1}{2\sqrt{1+\xi}}. \tag{26}$$

Writing again ξ instead of ξ_1, the result is

$$\frac{1}{A}\left(\frac{\partial \Psi}{\partial \tau_1}\frac{\partial V}{\partial \xi} - \frac{\partial \Psi}{\partial \xi}\frac{\partial V}{\partial \tau_1}\right) = \frac{\partial^2 V}{\partial \xi^2} - 2R_1\tau_1\frac{\partial^2 V}{\partial \xi \partial \tau_1} + (R_4\tau_1^2 + R_5)\frac{\partial^2 V}{\partial \tau_1^2} + R_3\tau_1\frac{\partial V}{\partial \tau_1}, \tag{27a}$$

$$\frac{1}{A}\left(\frac{\partial \Psi}{\partial \tau_1}\frac{\partial K}{\partial \xi} - \frac{\partial \Psi}{\partial \xi}\frac{\partial K}{\partial \tau_1}\right) - \frac{2K}{\xi^2+\eta^2}\left(\frac{\eta A'\tau_1 + \xi}{A}\frac{\partial \Psi}{\partial \tau_1} - \eta\frac{\partial \Psi}{\partial \xi}\right)$$

$$+4(\xi^2+\eta^2)V\left(\eta\frac{\partial V}{\partial \xi} + \frac{\xi - \eta A'\tau_1}{A}\frac{\partial V}{\partial \tau_1}\right) + \frac{4}{\xi^2+\eta^2}\left(\xi\frac{\partial K}{\partial \xi} + \frac{\eta - \xi A'\tau_1}{A}\frac{\partial K}{\partial \tau_1} - K\right)$$

$$= \frac{\partial^2 K}{\partial \xi^2} - 2R_1\tau_1\frac{\partial^2 K}{\partial \xi \partial \tau_1} + (R_4\tau_1^2 + R_5)\frac{\partial^2 K}{\partial \tau_1^2} + R_3\tau_1\frac{\partial K}{\partial \tau_1}, \tag{27b}$$

$$4K = \frac{\partial^2 \Psi}{\partial \xi^2} - 2R_1\tau_1 \frac{\partial^2 \Psi}{\partial \xi \partial \tau_1} + \left(R_4\tau_1^2 + R_5\right)\frac{\partial^2 \Psi}{\partial \tau_1^2} + R_3\tau_1 \frac{\partial K}{\partial \tau_1}, \tag{27c}$$

where

$$R = \frac{A'}{A}, \qquad R_3 = 2\frac{A'^2}{A^2} - \frac{A''}{A}, \qquad R_4 = \frac{A'^2}{A^2}, \qquad R_5 = \frac{1}{A^2}.$$

In order to keep the dependent variables finite for $\xi \to \infty$, we introduce

$$\Psi_1(\xi, \tau_1) = \frac{\Psi(\xi, \tau_1)}{(1+\xi)^2}, \qquad K_1(\xi, \tau_1) = \frac{K(\xi, \tau_1)}{(1+\xi)^3}. \tag{28}$$

Both Ψ_1 and K_1 vanish as $O(\xi^{-1/2})$ for $\xi \to \infty$. The equations for Ψ_1, K_1 and V become

$$\frac{(1+\xi)^2}{A}\left(\frac{\partial \Psi_1}{\partial \tau_1}\frac{\partial V}{\partial \xi} - \frac{\partial \Psi_1}{\partial \xi}\frac{\partial V}{\partial \tau_1}\right) - \frac{2(1+\xi)}{A}\Psi_1\frac{\partial V}{\partial \tau_1}$$

$$= \frac{\partial^2 V}{\partial \xi^2} - 2R_1\tau_1\frac{\partial^2 V}{\partial \xi \partial \tau_1} + \left(R_4\tau_1^2 + R_5\right)\frac{\partial^2 V}{\partial \tau_1^2} + R_3\tau_1\frac{\partial V}{\partial \tau_1}, \tag{29a}$$

$$\frac{(1+\xi)^2}{A}\left(\frac{\partial \Psi_1}{\partial \tau_1}\frac{\partial K_1}{\partial \xi} - \frac{\partial \Psi_1}{\partial \xi}\frac{\partial K_1}{\partial \tau_1}\right) + \frac{1+\xi}{A}\left\{3 - \frac{2(1+\xi)(\eta A'\tau_1 + \xi)}{\xi^2+\eta^2}\right\}K_1\frac{\partial \Psi_1}{\partial \tau_1}$$

$$+ \frac{2\eta(1+\xi)^2}{\xi^2+\eta^2}K_1\frac{\partial \Psi_1}{\partial \xi} - \frac{2(1+\xi)}{A}\Psi_1\frac{\partial K_1}{\partial \tau_1} + \frac{4\eta(1+\xi)}{\xi^2+\eta^2}\Psi_1 K_1$$

$$+ \frac{4(\xi^2+\eta^2)}{(1+\xi)^3}V\left(\eta\frac{\partial V}{\partial \xi} + \frac{\xi - \eta A'\tau_1}{A}\frac{\partial V}{\partial \tau_1}\right)$$

$$= \frac{\partial^2 K_1}{\partial \xi^2} - 2R_1\tau_1\frac{\partial^2 K_1}{\partial \xi \partial \tau_1} + \left(R_4\tau_1^2 + R_5\right)\frac{\partial^2 K_1}{\partial \tau_1^2} + \left(\frac{6}{1+\xi} - \frac{4\xi}{\xi^2+\eta^2}\right)\frac{\partial K_1}{\partial \xi}$$

$$+ \left\{R_3\tau_1 - \frac{6R_1\tau_1}{1+\xi} - \frac{4(\eta - \xi A'\tau_1)}{(\xi^2+\eta^2)A}\right\}\frac{\partial K_1}{\partial \tau_1}$$

$$+ \left\{\frac{6}{(1+\xi)^2} - \frac{12\xi}{(1+\xi)(\xi^2+\eta^2)} + \frac{4}{\xi^2+\eta^2}\right\}K_1, \tag{29b}$$

$$4(1+\xi)K_1 = \frac{\partial^2 \Psi_1}{\partial \xi^2} - 2R_1\tau_1\frac{\partial^2 \Psi_1}{\partial \xi \partial \tau_1} + \left(R_4\tau_1^2 + R_5\right)\frac{\partial^2 \Psi_1}{\partial \tau_1^2} + \frac{4}{1+\xi}\frac{\partial \Psi_1}{\partial \xi}$$

$$+ \left(R_3 - \frac{4R_1}{1+\xi}\right)\tau_1\frac{\partial \Psi_1}{\partial \tau_1} + \frac{2}{(1+\xi)^2}\Psi_1. \tag{29c}$$

One more transformation for each of the independent variables ξ and τ_1 has been performed. The infinite interval $\xi \in [0, \infty)$ has been transformed into $\sigma \in [0, 1]$ by the relation

$$\sigma = \frac{\xi}{\left(c + \sqrt{1+\xi}\right)^2} \quad \text{with} \quad c = 0.25. \tag{30}$$

Given σ, ξ follows from the quadratic equation

$$(1-\sigma)^2\xi^2 - 2\sigma\{(\sigma+1)c^2 - (\sigma-1)\}\xi + \sigma^2(c^2-1)^2 = 0.$$

For $\xi \to \infty$ it is easy to show that

$$1 - \sigma = 2c\xi^{-1/2} + O(\xi^{-1}).$$

Thus, Ψ_1 and K_1 vanish for $\sigma \to 1$ like $O(1-\sigma)$.

The transformation of $\tau_1 \in [0, 28]$ to $\mu \in [0, 1]$ is realized by

$$\tau_1 = 4\mu + 24\mu^4. \tag{31}$$

The advantage of this transformation is that an equidistant distribution of points in μ leads to a greater density of points for small than for large values of τ_1, which better corresponds to the behaviour of the dependent variables.

Derivatives to ξ in Eqns. (29) are now replaced by derivatives to σ using

$$\frac{\partial}{\partial\xi} = \frac{d\sigma}{d\xi}\frac{\partial}{\partial\sigma}, \qquad \frac{\partial^2}{\partial\xi^2} = \left(\frac{d\sigma}{d\xi}\right)^2\frac{\partial^2}{\partial\sigma^2} + \frac{d^2\sigma}{d\xi^2}\frac{\partial}{\partial\sigma}$$

with similar formulae for the change of τ_1-derivatives to μ-derivatives.

Next, the σ and μ-derivatives are replaced by central differences applying an equidistant grid with meshes h and k in the σ, μ unit square.

The boundary conditions are

$$\text{(i)} \quad \mu = 0, \qquad \Psi(\sigma, 0) = 0, \qquad V(\sigma, 0) = 0,$$

$$K_1(\sigma, 0) = R_5(\xi)\left(\frac{d\mu}{d\tau_1}\right)^2_{\tau_1=0} \frac{8\Psi_1(\sigma, k) - \Psi_1(\sigma, 2k)}{8k^2(1+\xi)}. \tag{32}$$

The last formula is the so-called plate condition, which follows from Eqn. (29c) using

$$\Psi_1 = \frac{\partial\Psi_1}{\partial\tau_1} = 0 \quad \text{for} \quad \tau_1 = 0.$$

The error in this formula is $O(k^2)$.

$$\text{(ii)} \quad \sigma = 0, \qquad \Psi_1(0, \mu) = 0, \qquad K_1(0, \mu) = 0.$$

The boundary condition $\partial V/\partial\xi = 0$ in ξ, η-coordinates, see Eqn. (25b), becomes in ξ, τ_1 coordinates

$$\frac{\partial V}{\partial \xi} - R_1(0)\tau_1 \frac{\partial V}{\partial \tau_1} = 0,$$

which leads to the discretized result

$$3V(0,\mu) = 4V(h,\mu) - V(2h,\mu) - \frac{\tau_1 R_1(0)\dfrac{d\mu}{d\tau_1}}{\left(\dfrac{d\sigma}{d\xi}\right)_{\xi=0}} \frac{h}{k}\{V(0,\mu+k) - V(0,\mu-k)\}. \tag{33}$$

Also this formula has an error which is quadratic in the mesh lengths.

(iii) $\mu = 1, \quad V(\sigma, 1) = 0, \quad K_1(\sigma, 1) = 0.$

Along this boundary the function Ψ_1 should be continued smoothly into region II.

(iv) $\sigma = 1, \quad \Psi_1(1,\mu) = 0, \quad K_1(1,\mu) = 0, \quad V(1,\mu) = V_0(\tau_1).$

Region II is limited by

$$\sigma = 0, \quad \sigma = 1, \quad \tau_1 = 2\sqrt{1+\xi}\cdot\eta_b = 28 \quad \text{and} \quad \eta_0 = 30.$$

In this region coordinates ξ_1 and λ, defined by

$$\xi_1 = \xi, \qquad \lambda = \frac{\eta - \eta_b(\xi)}{\eta_0 - \eta_b(\xi)} \tag{34}$$

are used.

Writing again ξ instead of ξ_1 and introducing at the same time Ψ_1 instead of Ψ in Eqn. (24c), this equation becomes

$$\frac{\partial^2\Psi_1}{\partial\xi^2} + \frac{2(\lambda-1)\eta_b'}{\eta_0-\eta_b}\frac{\partial^2\Psi_1}{\partial\xi\partial\lambda} + \frac{(\lambda-1)^2\eta_b'^2+1}{(\eta_0-\eta_b)^2}\frac{\partial^2\Psi_1}{\partial\lambda^2} + \frac{4}{1+\xi}\frac{\partial\Psi_1}{\partial\xi}$$
$$+\frac{\lambda-1}{\eta_0-\eta_b}\left(\frac{4\eta_b'}{1+\xi} + \eta_b'' + \frac{2\eta_b'^2}{\eta_0-\eta_b}\right)\frac{\partial\Psi_1}{\partial\lambda} + \frac{2}{(1+\xi)^2}\Psi_1 = 0. \tag{35}$$

The transformation from ξ to σ is again used.

The boundary conditions for region II are as follows:

(i) $\lambda = 0$: Smooth continuation of Ψ_1 toward region I,
(ii) $\sigma = 0$: $\Psi_1(0,\lambda) = 0$,
(iii) $\lambda = 1$: Smooth continuation of Ψ_1 toward region III,
(iv) $\sigma = 1$: $\Psi_1(1,\lambda) = 0$.

The σ and λ-derivatives in the equations are again replaced by central differences using an equidistant grid with meshes h and l in the σ, λ-unit square.

At $\lambda = 0$, the boundary between regions I and II, we also apply Eqn. (35). Derivatives to λ at $\lambda = 0$ can only be approximated by differences based upon the unequal meshes $\lambda = l$ and $\lambda = \lambda^{-}(\sigma)$, where $\lambda^{-}(\sigma)$ denotes the negative λ-value corresponding to the points in region I with $\mu = 1 - k$.

In *region III* we consider the equation

$$\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} = 0. \tag{36}$$

The boundary conditions are:

(i) $\xi = 0 : \Psi(0, \eta) = 0$,

(ii) $\eta = \eta_0$: Smooth continuation to the solution in region II.

(iii) In the ξ, η-plane the asymptotic behaviour in polar coordinates is

$$\Psi = P_1 r^{3/2}(\cos\tfrac{3}{2}\theta + \sin\tfrac{3}{2}\theta) + Q_1 r \cos\theta + S_1 r^{1/2}(\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta) + T_1\left(\frac{\pi}{2} - \theta\right). \tag{37}$$

Matching to region I means η small in such a way that $\xi^{1/2}\eta = \tau/2$ remains finite for $\xi \to \infty$. With

$$r = \xi\left(1 + \frac{\eta^2}{2\xi^2}\right), \qquad \cos\tfrac{1}{2}\theta = 1 - \frac{\eta^2}{8\xi^2}, \qquad \sin\tfrac{1}{2}\theta = \frac{\eta}{2\xi},$$

we obtain

$$\Psi = P_1(\xi^{3/2} + \tfrac{3}{4}\tau) + Q_1\xi + S_1\xi^{1/2} + \frac{\pi}{2}T_1. \tag{38}$$

From relations (11) and (14c) we find that both τ and τ_2 may be replaced by

$$\tau_1 - \tfrac{1}{2}\xi^{-1}\tau_1 + O(\xi^{-2})$$

and hence the asymptotic behaviour of Ψ in region I follows from (9c) and (18) as

$$\Psi = \xi^{3/2}\psi_0(\tau_1) - \tfrac{1}{2}\xi^{1/2}\tau_1\psi_0'(\tau_1) + \psi_3(\tau_1) + O(\xi^{-1/2}).$$

For $\tau_1 \to \infty$ (η small but unequal to 0), the exponential decrease of $\psi_0'(\tau_1)$ and the asymptotic behaviour of $\psi_3(\tau_1)$, given in Section 4, make that we can write

$$\Psi = \xi^{3/2}\psi_0(\infty) + \alpha\tau_1 + \beta$$

and hence, by comparison with (38) we have

$$P_1 = \psi_0(\infty), \qquad \alpha = \tfrac{3}{4}\psi_0(\infty), \qquad Q_1 = 0, \qquad S_1 = 0, \qquad T_1 = \frac{2}{\pi}\beta.$$

The asymptotic behaviour of Ψ in region III (and also in region II) is then obtained from (37) as

$$\Psi = \psi_0(\infty)\cdot r^{3/2}(\cos\tfrac{3}{2}\theta + \sin\tfrac{3}{2}\theta) + \beta\left(1 - \frac{2\theta}{\pi}\right).$$

Since the right-hand side is a harmonic function, we now introduce the harmonic function

$$\Psi_2(\xi, \eta) = \Psi(\xi, \eta) - \psi_0(\infty)\cdot r^{3/2}(\cos\tfrac{3}{2}\theta + \sin\tfrac{3}{2}\theta) - \beta\left(1 - \frac{2\theta}{\pi}\right), \tag{39}$$

which vanishes at infinity, satisfies $\Psi_2(0, \eta) = 0$ and should be in agreement with the solution of region II along the line $\eta = \eta_0$.

The solution of Ψ_2 in region III is obtained by a method due to Botta and Dijkstra [7], which uses a Green's function for the Laplace equation in the quarter-plane. This function is

$$G^{(1)}(P, Q) = \frac{1}{2\pi}\operatorname{Re}\log\frac{w^2 - w_1^2}{w^2 - w_1^{*2}}$$

where $P = (\lambda, \mu)$, $Q = (\lambda_1, \mu_1)$, $w = \lambda + i\mu$, $w_1 = \lambda_1 + i\mu_1$, $\lambda, \lambda_1, \mu, \mu_1, \geqslant 0$ and Re stands for "real part of".

When the harmonic function $\phi(Q)$ vanishes along the boundary $\mu = 0$ and at infinity, we obtain for ϕ in an arbitrary point P,

$$2\pi\phi(P) = -\int_0^\infty \phi(Q)\left[\frac{\partial}{\partial\mu_1}\operatorname{Re}\log\frac{w^2 - w_1^2}{w^2 - w_1^{*2}}\right]_{\mu_1 = 0} d\lambda_1, \qquad Q = (\lambda_1, 0),$$

$$\text{or}\quad 2\pi\phi(P) = -4\int_0^\infty \phi(Q)\operatorname{Im}\frac{\lambda_1}{w^2 - \lambda_1^2}\,d\lambda,$$

where Im denotes "imaginary part of".

For points P lying near the boundary $\mu = 0$ we modify the last integral as follows

$$2\pi\phi(P) = -4\int_0^\infty \{\phi(Q) - \phi(Q')\}\operatorname{Im}\frac{\lambda_1}{w^2 - \lambda_1^2}\,d\lambda_1 - 4\phi(Q')\int_0^\infty \operatorname{Im}\frac{\lambda_1}{w^2 - \lambda_1^2}\,d\lambda_1,$$

where $Q' = (\lambda, 0)$ is the projection of P on the boundary $\mu = 0$.

The last result can be reduced to the form

$$\pi\phi(P) = (\pi - 2\theta)\phi(Q') + 4\lambda\mu\int_0^\infty \{\phi(Q) - \phi(Q')\}\frac{\lambda_1 d\lambda_1}{(\lambda_1^2 - \lambda^2 + \mu^2)^2 + 4\lambda^2\mu^2},$$

where $\theta = \arg w = \tan^{-1}(\mu/\lambda)$.

Applying the last formula to the harmonic function Ψ_2 in the ξ, η-plane with values given along the line $\eta = \eta_0$, the result is

$$\pi\Psi_2(\xi, \eta) = \left(\pi - \tan^{-1}\frac{\eta - \eta_0}{\xi}\right)\Psi_2(\xi, \eta_0) + 4\xi(\eta - \eta_0)\int_0^{\infty}\{\Psi_2(\xi', \eta_0) - \Psi_2(\xi, \eta_0)\}$$

$$\times\frac{\xi' \mathrm{d}\xi'}{\{\xi'^2 - \xi^2 + (\eta - \eta_0)^2\}^2 + 4\xi^2(\eta - \eta_0)^2}. \tag{40}$$

The remaining integral is evaluated by aid of quadratic approximations of $\Psi_2(\xi', \eta_0) - \Psi_2(\xi, \eta_0)$ in each interval $\xi_{j-1} \leqslant \xi' \leqslant \xi_{j+1}$ with j odd. This allows analytic calculation of the integral in each interval, see [7]. The points ξ_j correspond to the mesh points obtained in the equidistant σ-distribution.

The points ξ, η in region III where Ψ_2 is calculated from (40) have the same ξ-values (or σ-values) as the points in regions I and II, while η is defined by (34) with $\lambda = 1 + l$, where l is the mesh length of the λ-distribution. Thus

$$\eta = \eta_0 + l\{\eta_0 - \eta_b(\xi)\}.$$

The values of $\Psi_2(\xi, \eta_0)$ follow from (39) by substitution of

$$\Psi(\xi, \eta) = (1 + \xi)^2\Psi_1(\xi, \eta_0), \qquad r = \sqrt{\xi^2 + \eta_0^2}, \qquad \theta = \tan^{-1}\frac{\eta_0}{\xi}.$$

Having obtained $\Psi_2(\xi, \eta)$ from (40), the corresponding value of $\Psi_1(\xi, \eta)$ is given by

$$\Psi_1(\xi, \eta) = \frac{1}{(1 + \xi)^2}\left\{\Psi_2(\xi, \eta) + \psi_0(\infty)r^{3/2}(\cos\tfrac{3}{2}\theta + \sin\tfrac{3}{2}\theta) + \beta\left(1 - \frac{2\theta}{\pi}\right)\right\}, \tag{41}$$

where $r = \sqrt{\xi^2 + \eta^2}$, $\theta = \tan^{-1}\eta/\xi$.

Then, an improved value of $\Psi_1(\xi, \eta_0)$ can be obtained from the same formulae as applied in region II, using the (σ, λ)-grid with

$$0 \leqslant \sigma \leqslant 1, \qquad 0 \leqslant \lambda \leqslant 1 + l.$$

7. Numerical calculations (practice)

The final solution for V, K_1 and Ψ_1 was obtained by application of the 5-point SOR-method.

One iteration step consisted of the following actions:

(i) calculation of V from Eqns. (29a) and (33) in all points of region I.

(ii) calculation of K_1 from Eqns. (29b) and (32) in all points of region I.

(iii) calculation of Ψ_1 from Eqns. (29c), (35) and (41) in all points of regions I and II as well as in the points $\lambda = 1 + l$. Equations (29) and (35) are used, of course, in their difference form. The relaxation factors were 0.6 for V, 0.5 for K_1 and 1.65 for Ψ_1.

The following initial values were taken:

$$\textit{region I}: V = V_0(\tau_1), \qquad K_1 = \frac{\xi + \xi^{5/2}}{(1+\xi)^3}\psi_0''(\tau_1), \qquad \Psi_1 = \frac{\xi + \xi^{3/2}}{(1+\xi)^2}\psi_0(\tau_1),$$

$$\textit{region II}: \Psi_1 = \frac{\xi + \xi^{3/2}}{(1+\xi)^2}\psi_0(\infty).$$

These values have the correct behaviour for $\xi \to 0$ and $\xi \to \infty$ as far as the main terms are concerned.

In order to check the solution, 3 different grids have been used. The coarsest grid is defined by

$$h = 0.1, \qquad k = 0.1, \qquad l = 0.2.$$

The two other grids were obtained by twice reducing all meshes by a factor 2.

8. Results

8.1. The azimuthal velocity V

In Fig. 2 curves of constant V-values in the X, Z-plane are presented. The curve denoted by $V = 1$ means in fact $V = 1 - 0.5 \times 10^{-5}$. According to Rogers and Lance [4] the value

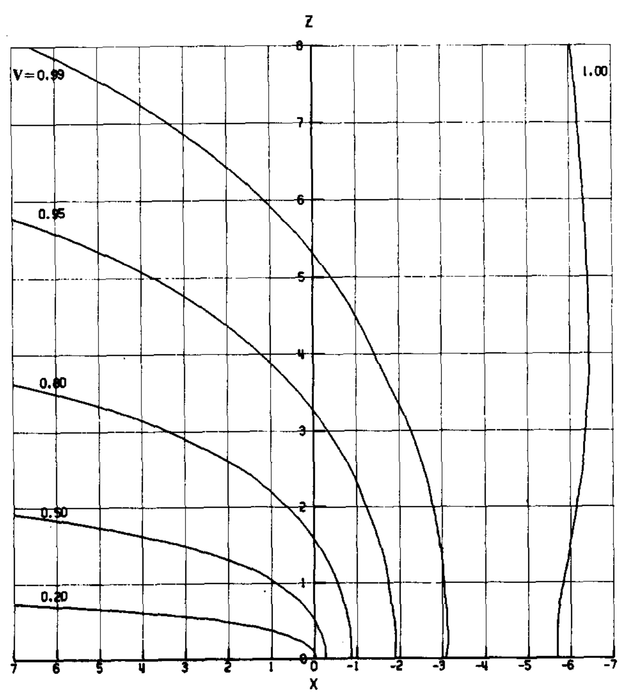

Figure 2. Lines of constant azimuthal velocity V.

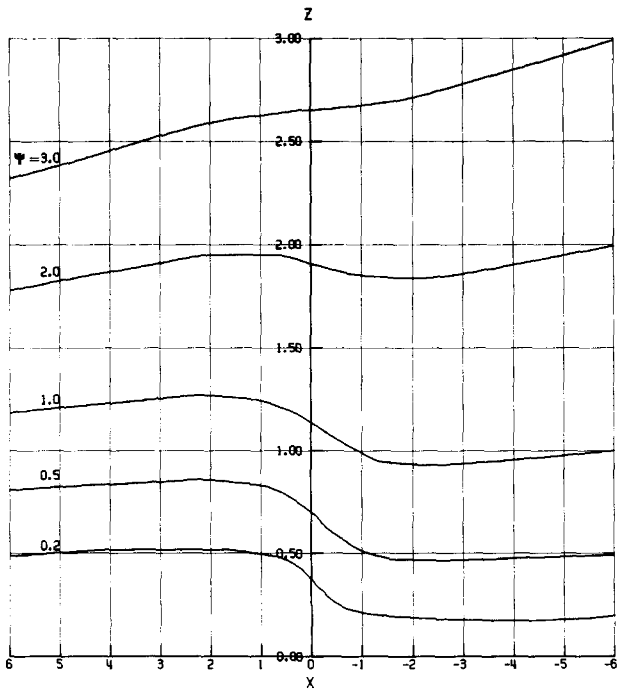

Figure 3. Lines of constant streamfunction Ψ.

$V = 1$ is approached for $\xi \to \infty$ as $e^{-\alpha\tau}$ with $\alpha = \frac{3}{4}\psi(\infty)$. However, the present calculations suggest that for small values of ξ (that is along the axis with negative values of X) the approach to $V = 1$ is no longer monotonous but contains a small oscillating factor. The corresponding overshoot is limited to 1.00020. Hence, it is not quite sure whether this is a real phenomenon or that it is due to a discretization error.

It follows from (4c) that for $\xi \to \infty$ (large positive values of X) the curves $V =$ constant have the asymptotic behaviour $X = cZ^4$.

8.2. The streamfunction Ψ

Figure 3 shows streamlines ($\Psi =$ constant) in the X, Z-plane. It is seen that for negative values of X (outside the disc) Ψ is proportional to Z, while for positive values of X (at the disc) Ψ is proportional to Z^2. Asymptotically for $X \to \infty$ the curves behave like $XZ^8 = c$.

The non-scaled streamfunction $\psi = Re^{-1}\Psi$ is $O(Re^{-1})$ in the Navier-Stokes region, see (2). However, for $X \to \infty$ Ψ becomes infinite like $X^{3/4} = Re^{1/2}(1-r)^{3/4}$. This matches the streamfunction ψ in the boundary-layer region which is $O(Re^{-1/2}(1-r)^{3/4})$.

8.3. The vorticity Γ

Curves of constant Γ are shown in Fig. 4. The curve $\Gamma = 0$ means in fact $\Gamma < 0.5 \times 10^{-5}$. Analogous to the situation with V, the exponential decrease toward $\Gamma = 0$ for large ξ contains for small ξ an oscillating factor leading to a maximum of $\Gamma = 0.00004$.

For $\xi \to \infty$, i.e. $X \to \infty$, Γ becomes infinite like $X^{1/4}$. For $Z = 0$ and negative X we have $\Gamma = 0$.

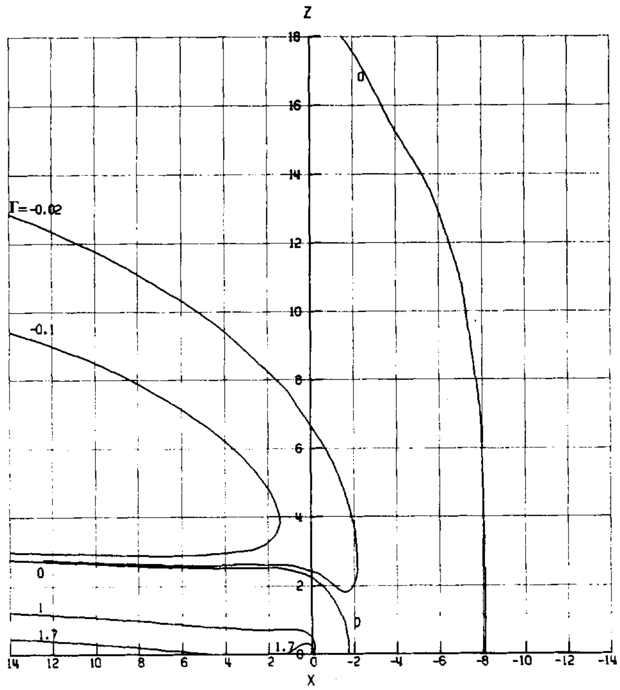

Figure 4. Lines of constant vorticity Γ.

At the origin the value of Γ is indefinite. All curves for positive Γ lead to the origin and are tangent to the negative X-axis at the origin. Approaching the origin along a straight line leads always to $\Gamma \to \infty$ (except along the negative X-axis).

The non-scaled vorticity γ is $O(Re^{1/3})$ in the Navier-Stokes region. For $X \to \infty$, Γ becomes infinite like $X^{1/4} = Re^{1/6}(1-r)^{1/4}$. This matches the vorticity γ in the boundary layer region, which is $O(Re^{1/2}(1-r)^{1/4})$.

8.4. The tangential shear stress τ_t^*

An asterisk denotes a physical quantity not made dimensionless. Then

$$\tau_t^* = \mu \frac{\partial v^*}{\partial z^*} = \mu\Omega \frac{\partial v}{\partial z} = \mu\Omega Re^{2/3} \frac{\partial V}{\partial Z} = \tfrac{1}{2}\mu\Omega Re^{2/3} \frac{1}{\xi} \frac{\partial V}{\partial Z}.$$

Since $\mu = \rho\nu = \rho a^2 \Omega Re^{-1}$, the shear stress becomes

$$\tau_t^* = \tfrac{1}{2}\rho a^2 \Omega^2 Re^{-1/3} \frac{1}{\xi} \frac{\partial V}{\partial \eta}.$$

Using Eqns. (26) and (31) to transform the derivative to η into a derivative to μ, we obtain

$$\tau_t^* = \tfrac{1}{4}\rho a^2 \Omega^2 Re^{-1/3} \frac{\sqrt{1+\xi}}{\xi} \frac{\partial V}{\partial \mu}.$$

Since the derivative in this formula should be taken at $\mu = 0$ and since $V(\sigma, 0) = 0$, the final result for the dimensionless shear stress is

$$\tau_t = \frac{1}{8k} Re^{-1/3} \frac{\sqrt{1+\xi}}{\xi} \{4V(\sigma, k) - V(\sigma, 2k)\}.$$

For $X \to \infty$ τ_t vanishes like $X^{-1/4} = Re^{-1/6}(1-r)^{-1/4}$. Hence, the shear stress in the boundary-layer region becomes infinite like $O(Re^{-1/2}(1-r)^{-1/4})$.

Figure 5 shows $Re^{1/3}\tau_t$ as function of X. For $X \to 0$ τ_t becomes infinite like $0.502 Re^{-1/3} X^{-1/2}$.

8.5. The radial shear stress τ_r^*

The radial shear stress is given by

$$\tau_r^* = u \frac{\partial u^*}{\partial z^*}.$$

We reduce

$$\tau_r^* = \mu \frac{\partial u^*}{\partial z^*} = \mu\Omega \frac{\partial u}{\partial z} = \mu\Omega Re^{1/3} \frac{\partial U}{\partial Z} = -\mu\Omega Re^{1/3} \frac{\partial^2 \Psi}{\partial Z^2} =$$

$$-\mu\Omega Re^{1/3}\Gamma = -\rho a^2 \Omega^2 Re^{-2/3} \frac{(1+\xi)^3}{\xi^2} K_1(\xi, 0).$$

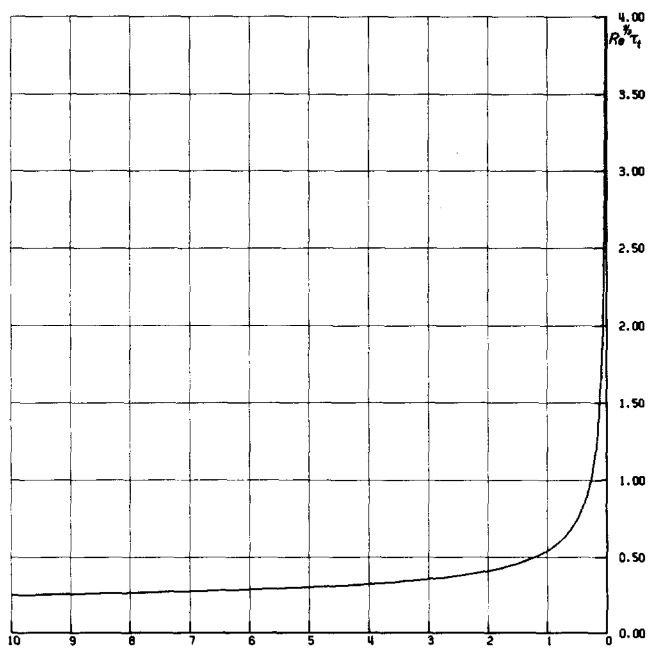

Figure 5. The tangential shear stress $Re^{1/3}\tau_t$ as function of X.

For $\xi \to 0$ K_1 vanishes like $O(\xi)$ according to (10), which implies that τ_r becomes infinite like $1.234 Re^{-2/3} X^{-1/2}$.

For $\xi \to \infty$ K_1 vanishes like $O(\xi^{-1/2})$ and, hence, τ_r becomes infinite like $\xi^{1/2} = X^{1/4} = Re^{1/6}(1-r)^{1/4}$. This matches the radial shear stress in the boundary layer which vanishes like $O(Re^{-1/2}(1-r)^{1/4})$.

Figure 6 shows $Re^{2/3}\tau_r$ as function of X. The negative values denote that the radial stress is directed toward the disc centre.

8.6. The torque M^* on the disc

The torque M^* is given by

$$M^* = 2\pi \int_0^a \tau_t^* r^{*2} \mathrm{d} r^*.$$

We are interested in the contribution of the Navier-Stokes region to the torque. Let this region extend from $r^* = r_0^*$ to $r^* = a$, where $a - r_0^* = aO(Re^{-2/3})$. Retaining only the main term in $O(Re)$ we have in this region

$$r^* = a, \qquad \mathrm{d} r^* = -aRe^{-2/3}\mathrm{d} X = -2aRe^{-2/3}\xi \mathrm{d}\xi.$$

while

$$\tau_t^* = \rho a^2 \Omega^2 \tau_t = \rho a^2 \Omega^2 Re^{-1/3} \frac{\sqrt{1+\xi}}{\xi} \frac{\partial V}{\partial \tau_1}.$$

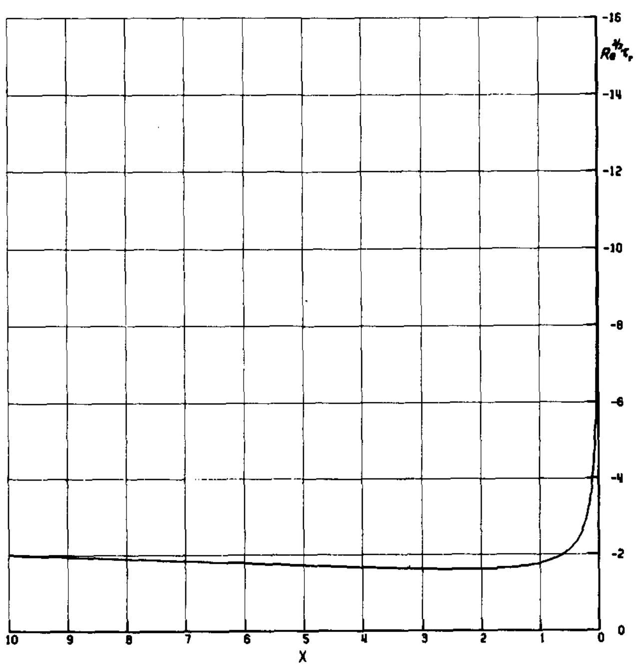

Figure 6. The radial shear stress $Re^{2/3}\tau_r$ as function of X.

Hence, the contribution of the Navier-Stokes region to M^* is equal to

$$4\pi\rho a^5\Omega^2 Re^{-1}\int_0^{\xi_0}\frac{\partial V}{\partial \tau_1}\sqrt{1+\xi}\,\mathrm{d}\xi,$$

where ξ_0 is the ξ-coordinate corresponding to the point r_0^*. For $\xi_0 \to \infty$ the integral diverges. This is due to the fact that boundary-layer theory leads to a torque $O(Re^{-1/2})$. According to boundary-layer theory the tangential shear stress is

$$\tau_t^* = \mu\Omega\frac{\partial v}{\partial z} = \mu\Omega\frac{\partial v}{\partial \tau}Re^{1/2}(1-r)^{-1/4} = \rho a^2\Omega^2 Re^{-1/2}\frac{\partial v}{\partial \tau}(1-r)^{-1/4}.$$

Integration over the Navier-Stokes region yields

$$\begin{aligned}
&2\pi\rho a^5\Omega^2 Re^{-1/2}\int_{r_0}^{1}\frac{\partial v}{\partial \tau}(1-r)^{-1/4}\mathrm{d}r\\
&\quad = 2\pi\rho a^5\Omega^2 Re^{-1}\int_0^{X_0}\frac{\partial V}{\partial \tau}X^{-1/4}\mathrm{d}X\\
&\quad = 4\pi\rho a^5\Omega^2 Re^{-1}\int_0^{\xi_0}\frac{\partial V}{\partial \tau}\xi^{1/2}\mathrm{d}\xi. \qquad (42)
\end{aligned}$$

The additional moment due to the fact that near the edge of the disc the Navier-Stokes equations should be used instead of the boundary-layer equations is

$$M_1^* = 4\pi\rho a^5\Omega^2 Re^{-1}\int_0^{\infty}\left\{\frac{\partial V}{\partial \tau_1}\sqrt{1+\xi} - V_0'(0)\sqrt{\xi}\right\}\mathrm{d}\xi.$$

This integral converges since

$$\frac{\partial V}{\partial \tau_1}\sqrt{1+\xi} = \frac{\partial V}{\partial \tau}\sqrt{\xi} \quad \text{and} \quad V = V_0(\tau) + O(\xi^{-2}) \quad \text{for} \quad \xi \to \infty, \qquad \text{see (22).}$$

The final result is

$$M_1^* = 4\pi\rho a^5\Omega^2 Re^{-1}\int_0^{1}\left\{\frac{4V(\sigma,k) - V(\sigma,2k)}{8k}\sqrt{1+\xi} - V_0'(0)\sqrt{\xi}\right\}\frac{\mathrm{d}\sigma}{\mathrm{d}\sigma/\mathrm{d}\xi}$$

or, after evaluation

$$M_1^* = 4.02\rho\alpha^5\Omega^2 Re^{-1}.$$

Since the integral in (42) diverges like $O(\xi_0^{3/2}) = O(X^{3/4}) = O(Re^{1/2})$, it is clear that the boundary-layer contribution to M^* is $O(Re^{-1/2})$.

8.7. The pressure p at the disc

According to the equation of motion given in Section 2 we have at the disc

$$\frac{\partial p}{\partial r} = Re^{-1}\frac{\partial^2 u}{\partial z^2}$$

or, in the Navier-Stokes region,

$$\frac{\partial p}{\partial X} - Re^{-2/3}\frac{\partial^2 U}{\partial Z^2} = Re^{-2/3}\frac{\partial^3 \Psi}{\partial Z^3} = Re^{-2/3}\frac{\partial \Gamma}{\partial Z}.$$

Integration leads to

$$p - p(0) = Re^{-2/3}\int_0^{X_0}\frac{\partial \Gamma}{\partial Z}\mathrm{d}X = Re^{-2/3}\int_0^{\xi_0}\frac{\partial \Gamma}{\partial \eta}\mathrm{d}\xi$$

$$= 2Re^{-2/3}\int_0^{\xi_0}\frac{\partial K}{\partial \tau}\sqrt{\xi}\,\frac{\mathrm{d}\xi}{\xi^2} = 2Re^{-2/3}\int_0^{\xi_0}\frac{\partial K_1}{\partial \tau_1}\,\frac{(1+\xi)^{7/2}}{\xi^2}\,\mathrm{d}\xi,$$

where $p(0)$ is the pressure exactly at the edge of the disc. It will later be shown that this pressure is finite.

Since for $\xi \to \infty$, $K_1 \sim O(\xi^{-1/2})$, the integral is divergent. As in Section 8.6, this is due to the fact that boundary-layer theory leads to pressure differences at the disc which are of a larger magnitude than $O(Re^{-2/3})$. In order to obtain a convergent integral we have to subtract the contribution of boundary-layer theory which can be done by subtracting the asymptotic expansion of $\partial K/\partial \tau$ for $\xi \to \infty$. Using (22), this expansion appears to be

$$\left.\frac{\partial K}{\partial \tau}\right|_{\tau=0} = \xi^{5/2}\psi_0'''(0) + \xi\psi_3'''(0) + \xi^{1/2}\psi_4'''(0) + 0(\xi^{-1/2}), \qquad \xi \to \infty.$$

Since $\psi_0'''(0) = -1$, $\psi_3'''(0) = 0$, $\psi_4'''(0) = 0$, the result for the pressure is

$$p - p(0) = 2Re^{-2/3}\int_0^{\xi}\left\{\frac{\partial K_1}{\partial \tau_1}\,\frac{(1+\xi)^{7/2}}{\xi^2} + \xi\right\}\mathrm{d}\xi \tag{43}$$

and this integral is convergent for $\xi \to \infty$.

The boundary-layer contribution is equal to

$$p - p(0)|_{b.l.} = -2Re^{-2/3}\int_0^{\xi}\xi\,\mathrm{d}\xi = -Re^{-2/3}\xi^2 = r - 1$$

or

$$p^* - p^*(0)|_{b.l.} = \rho a^2\Omega^2(r-1). \tag{44}$$

This is exactly in agreement with the pressure due to the centrifugal force, which is

$$p^* = \tfrac{1}{2}\rho a^2\Omega^2 r^2, \qquad p^*(0) = \tfrac{1}{2}\rho a^2\Omega^2$$

and hence leads, in first approximation again to (44).

It remains to show that $p(0)$ exists, which means that the integral in (43) should converge at the lower boundary. It has been shown in [7] that the next term in the expansion (10) of Ψ near the origin is

$$\Psi = B\left(\xi^3\eta^2 - \tfrac{1}{3}\xi\eta^4\right).$$

It follows from Eqns. (8) that near the origin this second term also satisfies

$$\Delta\Psi = 4(\xi^2+\eta^2)\Gamma \quad \text{and} \quad \Delta\Gamma = 0.$$

Hence $\Gamma = \frac{1}{2}\xi B$ and $K = \frac{1}{2}\xi(\xi^2+\eta^2)B$.

Then for $\xi \to 0$, $\partial K/\partial\tau_1 = \frac{1}{2}\partial K/\partial\eta = \frac{1}{2}\xi\eta B$, which vanishes for $\eta = 0$. Hence $\partial K/\partial\tau_1$ is at most $O(\xi^2)$ for $\xi \to 0$ and this guarantees the convergence of the integral in (43) at the lower boundary.

Due to the inaccuracies in the numerical calculation of the integrand in (43), both for small and for large values of ξ, it appeared to be impossible on the basis of the performed calculations to present reliable results for the pressure term of $O(Re^{-2/3})$.

8.8. The velocity U outside the disc in the plane Z = 0

The plane $Z = 0$ outside the disc corresponds to $\xi = 0$. We have

$$U = -\frac{\partial\Psi}{\partial Z} = -\frac{1}{2\eta}\left(\frac{\partial\Psi}{\partial\xi}\right)_\eta = -\frac{1}{2\eta}\left\{\left(\frac{\partial\Psi}{\partial\xi}\right)_{\tau_1} - \tau_1 R_1(\tau_1)\frac{\partial\Psi}{\partial\tau_1}\right\}.$$

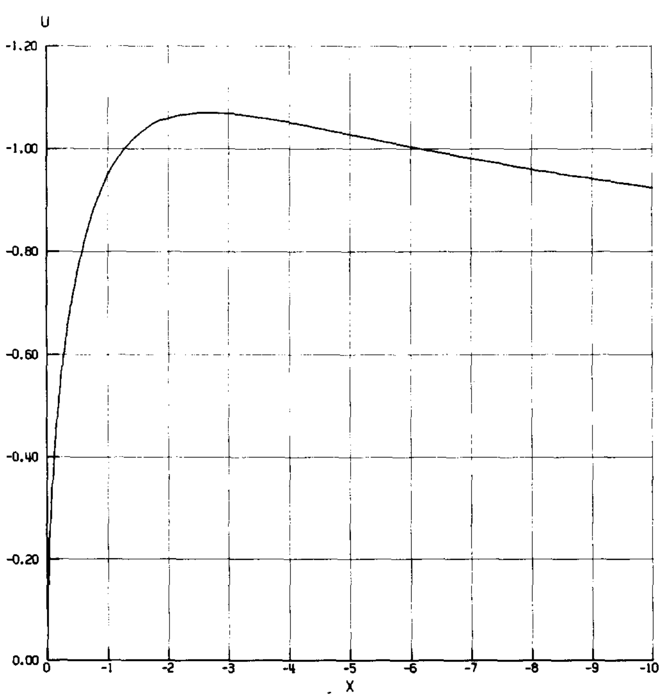

Figure 7. The velocity U outside the disc in the plane $Z = 0$.

For $\xi = 0$ holds $\partial\Psi_1/\partial\tau_1 = 0$ and hence,

$$U = -\frac{1}{2\eta}\left(\frac{\partial\Psi}{\partial\xi}\right)_{\tau_1} = -\frac{1}{2\eta}\left\{(1+\xi)^2\frac{\partial\Psi_1}{\partial\xi} + 2(1+\xi)\Psi_1\right\} = -\frac{1}{2\eta}\frac{\partial\Psi_1}{\partial\xi}.$$

Using (30) for calculating $d\sigma/d\xi = 1/(1+c)^2$ if $\xi = 0$, we find

$$U = -\frac{4\Psi_1(h,\eta) - \Psi_1(2h,\eta)}{4h\eta(1+c)^2}.$$

Thus

$$u^* = -a\Omega Re^{-1/3}\frac{4\Psi_1(h,\eta) - \Psi_1(2h,\eta)}{4h\eta(1+c)^2}.$$

Figure 7 shows $Re^{-1/3}u$ as function of X.

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